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The torque on an axisymmetric body in asymmetric rotational flow

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Abstract. The torque acting on an axisymmetric solid body which rotates about an axis perpendicular to its axis of symmetry is determined by a method which relaxes one of the three boundary conditions on the body. The method leads to an axisymmetric boundary value problem whose solution uniquely determines the torque. Illustrations including the prolate and oblate ellipsoid and limiting results relating to the sphere and thin circular disk provide corroboration with known results.

1. Introduction

A problem of fundamental importance in many engineering applications of the theory of suspensions in sedimentation or aerosols is the determination of the Stokes resistance of a small particle in motion in a fluid which is in general undergoing shear. An example would be the transport of solid particles in a pressure driven flow through a tube or channel. The theoretical problems which model these applications are naturally problems of great mathematical complexity involving in general particle-particle and particle-wall interactions as well as the basic particle-fluid interaction. The most fundamental of these problems is the latter.

If an isolated body rotates in unbounded fluid at rest at infinity and the Reynolds Number for the flow is sufficiently small for the Stokes equations of motion to apply, then the body experiences a torque which is equal and opposite to that acting on any other enveloping surface, as demonstrated in Appendix I. By choosing the enveloping surface to be a large sphere with centre at the origin, it is clear that the torque on the body arises from the rotlet singularity term in the far field asymptotic expansion of the velocity. The rotlet singularity represents the continuous application of torque applied at a point interior to the body and situated along its axis of rotation. For an axisymmetric body with fore-aft symmetry about a plane through its centre perpendicular to the axis of symmetry, this point of application of the rotlet singularity would be its centre. The velocity field for a rotlet is $O(r^{-2})$ as $r \to \infty$, with r measuring distance from the origin, and terms $O(r^{-n})$ for $n \geq 3$ in the asymptotic expansion of the velocity cannot contribute to the resultant torque acting on the body.

For rotation about the axis of symmetry, the problem was investigated by Jeffery [1] who showed that the pressure field is constant and the fluid velocity consists of one component orthogonal to the azimuthal plane. The solution for this velocity component was found explicitly for a number of body geometries. Chwang and Wu [2] approached this problem from a different viewpoint and showed how exact solutions for rotating bodies can be constructed by considering suitably chosen distributions of rotlets along the axis of symmetry. Their work corroborates that of Jeffery for the torque coefficient for prolate or oblate ellipsoids. When the body rotates about an axis which is not the axis of symmetry, there is a scarcity of exact solutions. Slender body theory, applicable when the axial dimension of the body greatly exceeds any transverse dimension, provides approximate solutions for such bodies, as is demonstrated by Batchelor [3] and Cox [4] for example. An exact solution was determined by Edwardes [5] for slow rotation of a general ellipsoid about a principal axis. The torque on an ellipsoid of revolution rotating about its axis of symmetry is found to agree with that of Jeffery if a numerical factor, incorrectly calculated as 32/5, is replaced by the correct value 16/3. Brenner [6] examined the limiting case of a circular disk and pointed out that the torque is invariant about any axis of rotation through the centre of the disk. This remarkable property is of course also possessed by the sphere. It is further shown by Brenner to be a property possessed by some other bodies such as a cube. It is worth noting that no similar drag invariance property exists for the translating disk. Jeffery [7] obtained an exact solution for a general ellipsoid in a linear shear flow and properties of this solution have been extensively studied by Hinch and Leal [8].

The asymmetric rotation problem is evidently more complicated analytically because in addition to a non-vanishing pressure field there are three velocity components which must now be determined. As demonstrated for instance by Lamb [12] and further elaborated in Appendix II, the general solution of the Stokes equations involves the evaluation of three quasi-harmonic scalar functions which through their coupling cannot be determined sequentially. As stated earlier, the torque acting on the body depends *only* on the strength of the rotlet singularity in the asymptotic expansion of the velocity field at large distances from the body. The purpose of this paper is to demonstrate that the determination of this rotlet strength does not however, require the determination of the complete flow field. We show that by relaxing one of the three boundary conditions on the body, the rotlet strength, and therefore the torque acting on the body, is uniquely specified on solving a tractable boundary value problem for an *axisymmetric* biharmonic function. The complete flow field would be the addition of this 'relaxed' flow field to a complementary flow field whose rate of decay at infinity is $O(r^{-3})$ and consequently cannot contribute to the torque. The analysis is presented in a general form appropriate to a wide class of axisymmetric bodies. Illustrative examples are the prolate and oblate ellipsoids where the torques are shown to agree with Edwardes if the numerical factor in his work is corrected. The limiting configuration of sphere and circular disk are also considered and our results corroborate those of Brenner. The analysis may also be applied directly to the case when the body is at rest in a fluid which, in the absence of the body, is in rigid body rotation. Clearly the torque acting on a body at rest is equal and opposite to that acting on a body rotating in fluid at rest.

2. The equations and method of solution

The fluid motion to be considered is the three-dimensional Stokes or creeping flow of unbounded fluid in the presence of an axisymmetric solid of revolution for which the z-axis is the axis of symmetry. It is also supposed the meridian boundary curve of the solid is Γ and is symmetric about the plane $z = 0$. The flow is forced by the rotation of the body about the y-axis in fluid at rest at infinity. Consequently the boundary value problem for the fluid velocity $\mathbf{q}$ and pressure p may be stated in non-dimensional form as follows:

$$0 = -\text{grad}\, p + \nabla^2 \mathbf{q}\,, \qquad \text{div}\,\mathbf{q} = 0\,, \tag{1}$$

$$\mathbf{q} \to 0 \quad \text{as } r \to \infty\,, \qquad \mathbf{q} = [\hat{\mathbf{j}} \times \mathbf{r}] \quad \text{on } \Gamma\,. \tag{2}$$

In fact the asymptotic expansion of the fluid velocity for large r can be developed following Happel and Brenner (9). We find that as $r \to \infty$,

$$\mathbf{q} \sim k \frac{[\hat{\mathbf{j}} \times \mathbf{r}]}{r^3} + \frac{3mxz\mathbf{r}}{r^5} + O(r^{-3}) , \tag{3}$$

$$p \sim \frac{6mxz}{r^5} + O(r^{-5}) , \tag{4}$$

where k and m are constants. The first and second terms on the right-hand side of (3) are, respectively, the velocity due to the rotlet and stresslet singularities. The rotlet has at most a constant pressure field. Since the torque acting on the body may be determined directly from the torque acting on a large sphere, it is easy to show that the contribution from the stresslet is zero while the contribution from the rotlet is $8\pi\mu k\hat{\mathbf{j}}$, with μ the coefficient of viscosity. On letting $r \to \infty$, all other terms in the expansions of $\mathbf{q}$ and p give zero contribution. Thus, the torque acting on the body is

$$\mathbf{G} = -8\pi\mu k\hat{\mathbf{j}} .$$

If $x = \rho \cos \phi$, $y = \rho \sin \phi$, $z = z$ define cylindrical polar coordinates, the boundary condition (2), the asymptotic conditions (3) and (4), together with equations (1) imply that the dependence of $\mathbf{q}$ and p on the azimuthal angle ϕ must be of the form

$$\mathbf{q} = q_z(\rho, z)\hat{\mathbf{k}} \cos \phi + q_\rho(\hat{\rho}, z)\hat{\boldsymbol{\rho}} \cos \phi + q_\phi(\rho, z)\hat{\boldsymbol{\phi}} \sin \phi , \tag{5}$$

$$p = \Pi(\rho, z) \cos \phi . \tag{6}$$

It is therefore only necessary to consider a solution of (1) for the first order Fourier component in ϕ and a solution of this form, involving three independent scalar functions, is shown in Appendix II to be

$$\mathbf{q} = \operatorname{curl}(C\hat{\boldsymbol{\phi}} \cos \phi) + \operatorname{grad}\left(\frac{B}{\rho} \cos \phi\right) + \operatorname{curl}\left(\frac{A}{\rho} \hat{\mathbf{k}} \sin \phi\right) , \tag{7}$$

where A, B, C are solutions of

$$L_{-1}(A) = L_{-1}(B) = L_{-1}(C) = 0 , \qquad L_{-1} \equiv \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial}{\partial \rho} . \tag{8}$$

It therefore follows

$$q_z = \frac{\partial C}{\partial \rho} + \frac{C}{\rho} + \frac{1}{\rho}\frac{\partial B}{\partial z} , \qquad q_\rho = -\frac{\partial C}{\partial z} + \frac{1}{\rho}\frac{\partial B}{\partial \rho} + \frac{(A - B)}{\rho^2} ,$$

$$q_\phi = -\frac{1}{\rho}\frac{\partial A}{\partial \rho} + \frac{(A - B)}{\rho^2} , \tag{9}$$

and the pressure is given by

$$p = -\frac{2}{\rho}\frac{\partial C}{\partial z} \cos \phi .$$

Now if A, B, C are given by the particular solutions

$$A = -\frac{kz}{r}, \qquad B = -\frac{kz}{r}, \qquad C = \frac{m\rho^2}{r^3}, \tag{10}$$

with k and m constants, then the corresponding form of $\mathbf{q}$ is given by

$$\mathbf{q} = \frac{3mxz\mathbf{r}}{r^5} + \frac{k[\hat{\mathbf{j}} \times \mathbf{r}]}{r^3}. \tag{11}$$

Conversely since

$$\mathbf{q}_1 = \operatorname{curl}(C\hat{\boldsymbol{\phi}} \cos\phi), \qquad \mathbf{q}_2 = \operatorname{grad}\left(\frac{B}{\rho}\cos\phi\right), \qquad \mathbf{q}_3 = \operatorname{curl}\left(\frac{A}{\rho}\hat{\mathbf{k}}\sin\phi\right), \tag{12}$$

are linearly independent solutions of the Stokes equations, it follows that if $\mathbf{q}$ is represented by equation (11) then the most general forms of A, B, C are represented by equations (10).

Returning to the general velocity field, we define an axisymmetric harmonic function D by the Stokes–Beltrami equations

$$\frac{1}{\rho}\frac{\partial B}{\partial z} = \frac{\partial D}{\partial \rho}, \qquad -\frac{1}{\rho}\frac{\partial B}{\partial \rho} = \frac{\partial D}{\partial z}, \qquad L_1(D) \equiv \left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2}\right]D = 0, \tag{13}$$

and it is also expedient to set $C = \rho\, \partial E/\partial \rho$ where E is also an axisymmetric harmonic satisfying $L_1(E) = 0$. It is noted that in the particular case when $C = m\rho^2/r^3$, then $E = -m/r$. Writing $\chi = C + D + E$, the velocity components can be expressed as

$$q_z = \frac{\partial \chi}{\partial \rho}, \qquad q_\rho = -\frac{\partial \chi}{\partial z} + \frac{\partial E}{\partial z} + \frac{(A - B)}{\rho^2}, \qquad q_\phi = -\frac{1}{\rho}\frac{\partial A}{\partial \rho} + \frac{(A - B)}{\rho^2}. \tag{14}$$

The function χ is a solution of the repeated operator equation $L_1^2(\chi) = 0$ and is consequently an *axisymmetric* biharmonic. The solution can be represented by the decomposition formula [10]

$$\chi = u^{(1)} + v^{(-1)}, \qquad L_1(u^{(1)}) = 0, \qquad L_{-1}(v^{(-1)}) = 0. \tag{15}$$

It is further observed that both $\partial E/\partial z$ and $\rho^{-1}\,\partial A/\partial \rho$ are axisymmetric harmonics. By writing

$$\psi = \frac{\partial E}{\partial z} + \frac{1}{\rho}\frac{\partial A}{\partial \rho}, \tag{16}$$

then $L_1(\psi) = 0$ and we see that equations (14) imply that

$$q_z = \frac{\partial \chi}{\partial \rho}, \qquad q_\rho - q_\phi = -\frac{\partial \chi}{\partial z} + \psi. \tag{17}$$

On the boundary of the body, equations (17) give

$$\frac{\partial \chi}{\partial \rho} = -\rho, \qquad \frac{\partial \chi}{\partial z} = -2z + \psi. \tag{18}$$

It is therefore necessary to find axisymmetric functions ψ and χ such that

$$L_1\psi = L_1^2\chi = 0 , \tag{19}$$

which satisfy equations (18) on the body and the asymptotic conditions

$$\psi \sim \frac{(m+k)z}{r^3} + O\left(\frac{1}{r^2}\right), \tag{20}$$

$$\chi \sim \frac{(k-m)}{r} + \frac{m\rho^2}{r^3} + O\left(\frac{1}{r}\right), \tag{21}$$

as $r \to \infty$. In general, this boundary value problem determines the constants k and m, and therefore the torque acting on the body. It will be noticed that the velocity and pressure fields $\mathbf{q}^{(r)}$ and $p^{(r)}$ resulting from this solution do not in general satisfy the third non-slip condition involving $q_\rho + q_\phi$. The true velocity and pressure fields $\mathbf{q}^{(t)}$ and $p^{(t)}$ are evidently given by

$$\mathbf{q}^{(t)} = \mathbf{q}^{(r)} + \mathbf{q}^{(c)} ,$$

$$p^{(t)} = \mathbf{p}^{(r)} + p^{(c)} ,$$

where the complementary fields $\mathbf{q}^{(c)}$ and $p^{(c)}$ ensure the satisfaction of all three boundary conditions. Since $\mathbf{q}^{(r)}$ and $p^{(r)}$ determine correctly the rotlet and stresslet strengths in the far field, the fields $\mathbf{q}^{(c)}$ and $p^{(c)}$ give no contribution to the torque acting on the body.

It is assumed that the equation of the body is expressible by the conformal mapping

$$z + i\rho = f(\alpha + i\beta) ,$$

with $f'(\alpha + i\beta) \neq 0$ in the flow region and $\alpha = \alpha_0 = \text{constant}$ defines the meridian curve Γ. The function χ which satisfies $L_1^2\chi = 0$ is represented by the decomposition formula

$$\chi = u^{(1)} + v^{(-1)} . \tag{22}$$

The axisymmetric harmonic $u^{(1)}$ and the axisymmetric stream function $v^{(-1)}$ satisfy the equations in α, β coordinates given by

$$\frac{\partial}{\partial\alpha}\left(\rho\frac{\partial u^{(1)}}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\rho\frac{\partial u^{(1)}}{\partial\beta}\right) = 0 , \tag{23}$$

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{\rho}\frac{\partial v^{(-1)}}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{1}{\rho}\frac{\partial v^{(-1)}}{\partial\beta}\right) = 0 . \tag{24}$$

The inner and outer boundary conditions are given by

$$\left.\begin{aligned} \frac{\partial\chi}{\partial\alpha} &= (\psi - 2z)\frac{\partial z}{\partial\alpha} - \rho\frac{\partial\rho}{\partial\alpha} , \\ \frac{\partial\chi}{\partial\beta} &= (\psi - 2z)\frac{\partial z}{\partial\beta} - \rho\frac{\partial\rho}{\partial\beta} , \end{aligned}\right\} \tag{25}$$

on Γ and

$$\left.\begin{aligned}\psi &\sim \frac{(m+k)z}{r^3} + O\left(\frac{1}{r^2}\right),\\ \chi &\sim \frac{(k-m)}{r} + \frac{m\rho^2}{r^2} + O\left(\frac{1}{r}\right),\end{aligned}\right\} \tag{26}$$

as $r \to \infty$. Although the torque is independent of m, it is necessary to retain m in the general formulation of the boundary value problem in order to determine k uniquely. In the next sections we give some examples of the application of the general theory.

3. The sphere

In spherical polar coordinates (r, θ) the appropriate forms for $u^{(1)}$ and $v^{(1)}$ are given by

$$u^{(1)} = -\frac{(k-m)}{r} + \frac{F(3\cos^2\theta - 1)}{r^3}, \qquad v^{(-1)} = \frac{m \sin^2\theta}{r}, \tag{27}$$

implying

$$\chi = \frac{(k-m)}{r} + \frac{m\sin^2\theta}{r} + \frac{F(3\cos^2\theta - 1)}{r^3}, \tag{28}$$

where m, k, F are constants. The boundary conditions are

$$\frac{\partial\chi}{\partial r} = (m-k-1)\cos^2\theta - 1, \qquad \frac{\partial\chi}{\partial\theta} = (1-m-k)\sin\theta\cos\theta, \tag{29}$$

at $r = 1$ and

$$\chi \sim \frac{(k-m)}{r} + \frac{m\sin^2\theta}{r} \quad \text{as } r \to \infty. \tag{30}$$

It is a routine procedure to show that $F = m = 0$, and $k = 1$, producing the known correct expression $\mathbf{G} = -8\pi\mu\hat{\mathbf{j}}$ for the torque.

4. The oblate ellipsoid

Oblate ellipsoidal coordinates are defined by

$$z + i\rho = \sinh(\alpha + i\beta), \tag{31}$$

or equivalently

$$z = st, \qquad \rho = (s^2+1)^{1/2}(1-t^2)^{1/2}, \tag{32}$$

where $s = \sinh\alpha$ and $t = \cos\beta$. The surface $\alpha = \alpha_0$ or $s = s_0$ corresponds to the oblate ellipsoid whose equation is

$$\left(\frac{z}{\sinh\alpha_0}\right)^2 + \left(\frac{\rho}{\cosh\alpha_0}\right)^2 = 1. \tag{33}$$

With $\chi = u^{(1)} + v^{(-1)}$, the functions $u^{(1)}$ and $v^{(-1)}$ satisfy

$$\frac{\partial}{\partial s}\left[(s^2+1)\frac{\partial u^{(1)}}{\partial s}\right] + \frac{\partial}{\partial t}\left[(1-t^2)\frac{\partial u^{(1)}}{\partial t}\right] = 0 , \tag{34}$$

$$(s^2+1)\frac{\partial^2 v^{(-1)}}{\partial s^2} + (1-t^2)\frac{\partial^2 v^{(-1)}}{\partial t^2} = 0 . \tag{35}$$

Suitable functional forms for $u^{(1)}$ and $v^{(-1)}$ are

$$u^{(1)} = (k-m)q_0(s) + 2Fq_2(s)P_2(t) , \tag{36}$$

$$v^{(-1)} = \tfrac{3}{2}mq_1'(s)(s^2+1)(1-t^2) , \tag{37}$$

where F is a constant and $q_n(s) = i^{n+1}Q_n(is)$, with the prime denoting differentiation with respect to s. From Morse and Feshbach [11],

$$\left.\begin{aligned} q_0(s) &= \tan^{-1}(1/s) , & q_1(s) &= 1 - s\tan^{-1}(1/s) , \\ q_2(s) &= \tfrac{1}{2}[(3s^2+1)\tan^{-1}(1/s) - 3s] , & P_2(t) &= \tfrac{1}{2}(3t^2-1) . \end{aligned}\right\} \tag{38}$$

Thus

$$\chi = (k-m)q_0(s) - \tfrac{3}{2}mq_1'(s)(s^2+1)(1-t^2) + 2Fq_2(s)P_2(t) , \tag{39}$$

and since $L_1\psi = 0$, a suitable form for ψ is

$$\psi = 3(k+m)q_1(s)t . \tag{40}$$

Furthermore, as s and $r \to \infty$,

$$q_0(s) \sim s^{-1} \sim r^{-1} , \qquad 3q_1(s) \sim s^{-2} \sim r^{-2} , \qquad \tfrac{15}{2}q_2(s) \sim s^{-3} \sim r^{-3} .$$

It is therefore clear that the choice of χ and ψ satisfy the asymptotic conditions in the far field as $r \to \infty$. There remains the two boundary conditions on the body to be satisfied. These require

$$\left.\begin{aligned} \frac{\partial \chi}{\partial s} &= 3(k+m)q_1(s_0)t^2 - s_0(t^2+1) , \\ \frac{\partial \chi}{\partial t} &= 3(k+m)q_1(s_0)s_0t + t(1-s_0^2) , \end{aligned}\right\} \tag{41}$$

when $s = s_0$. With χ given by (39), these equations give rise to three equations for the constants m, k and F, namely

$$q_0'(s_0)k - [3q_1(s_0) + q_0'(s_0)]m - q_2'(s_0)F + s_0 = 0 , \tag{42}$$

$$3q_1'(s_0)k - 3q_2'(s_0)F - s_0 = 0 , \tag{43}$$

$$3s_0q_0(s_0)k - 3[s_0q_1(s_0) - (s_0^2+1)q_0'(s_0)]m - 6q_2(s_0)F + 1 - s_0^2 = 0 , \tag{44}$$

and since

$$3q_1(s_0) + q_0'(s_0) = -q_2'(s_0)\,, \qquad s_0 q_1(s_0) - (s_0^2+1)q_1'(s_0) = \tan^{-1}(1/s_0)\,,$$

equations (42), (43) and (44) simplify to

$$\left.\begin{aligned} &k - m + 4f = 0\,,\\ &k - \sigma_1 F = s_0/3q_1(s_0)\,,\\ &m + \tau_1 F = -1/3\tan^{-1}(1/s_0)\,,\end{aligned}\right\} \tag{45}$$

with $\sigma_1 = q_2'(s_0)/q_1(s_0)$ and $\tau_1 = (s_0 q_2'(s_0) - 2q_2(s_0))/\tan^{-1}(1/s_0)$. The coefficient k is readily found to be

$$k = \tfrac{2}{3}(2s_0^2+1)[(1-s_0)^2\tan^{-1}(1/s_0) + s_0]^{-1}\,, \tag{46}$$

and hence the torque acting on an oblate ellipsoid rotating with unit angular velocity about the minor principal axis coincident with the y-axis is

$$\mathbf{G} = -\tfrac{16}{3}\pi\mu\hat{\mathbf{j}}(2s_0^2+1)[(1-s_0^2)\tan^{-1}(1/s_0) + s_0]^{-1}\,. \tag{47}$$

Edwardes' expression for the torque is expressed in terms of elliptic integrals which may be written in terms of simple functions when the ellipsoid has rotational symmetry. For such bodies, Edwardes' formula for the torque agrees with (47) when an incorrect constant factor of 32/5 is replaced by 16/3. The limiting case $s_0 \to 0$ corresponds to a circular disk of radius unity. Equation (47) then yields $\mathbf{G} = -32\mu\hat{\mathbf{j}}/3$ which agrees with the result reported by Brenner (1963) and available in Happel and Brenner [9]. The case $s_0 \gg 1$ corresponds to a large sphere of radius s_0. Equation (47) then yields $\mathbf{G} = -8\pi\mu s_0^3\hat{\mathbf{j}}$, which is the well known Kirchhoff formula.

If the oblate ellipsoid rotates with angular velocity $\Omega\hat{\mathbf{j}}$ and the minor semi-axis length is a, the torque acting on the body is then $\mathbf{G} = -8\pi\Omega k a^3\hat{\mathbf{j}}/(s_0^2+1)^{3/2}$.

The values of the other solutions of equations (45) are

$$m = \tfrac{2}{3}[(s_0^2-1)\tan^{-1}(1/s_0) - s_0]^{-1}\,, \tag{48}$$

$$F = \tfrac{1}{3}(s_0^2+1)[(s_0^2-1)\tan^{-1}(1/s_0) - s_0]^{-1}\,. \tag{49}$$

The constant m gives the strength of the stresslet singularity at infinity when the oblate ellipsoid rotates about the y-axis.

5. The prolate ellipsoid

In a similar way the conformal transformation for a prolate ellipsoid is defined by

$$z + i\rho = \cosh(\alpha + i\beta)\,, \tag{50}$$

or equivalently

$$z = \cosh\alpha\cos\beta\,, \qquad \rho = \sinh\alpha\sin\beta\,. \tag{51}$$

The surface $\alpha = \alpha_0$, corresponds to the prolate ellipsoid expressed by the equation

$$\left(\frac{z}{\cosh \alpha_0}\right)^2 + \left(\frac{\rho}{\sinh \alpha_0}\right)^2 = 1 . \tag{52}$$

Again the function χ can be written as $\chi = u^{(1)} + v^{(-1)}$, where $u^{(1)}$ and $v^{(-1)}$ satisfy the equations

$$\frac{\partial}{\partial s}\left[(s^2-1)\frac{\partial u^{(1)}}{\partial s}\right] + \frac{\partial}{\partial t}\left[(1-t^2)\frac{\partial u^{(1)}}{\partial t}\right] = 0 , \tag{53}$$

$$(s^2-1)\frac{\partial^2 v^{(-1)}}{\partial s^2} + (1-t^2)\frac{\partial^2}{\partial t^2} v^{(-1)} = 0 , \tag{54}$$

with $s = \cosh \alpha$, $t = \cos \beta$. Suitable forms for $u^{(1)}$ and $v^{(-1)}$ are given by

$$u^{(1)} = (k-m)Q_0(s) + 2FQ_2(s)P_2(t) , \tag{55}$$

$$v^{(-1)} = -\tfrac{3}{2}mQ_1'(s)(s^2-1)(1-t^2) , \tag{56}$$

where F is a constant. Solutions for χ and ψ are therefore

$$\chi = (k-m)Q_0(s) - \tfrac{3}{2}Q_1'(s)(s^2-1)(1-t^2) + 2FQ_2(s)P_2(t) , \tag{57}$$

$$\psi = 3(k+m)Q_1(s)t , \tag{58}$$

to satisfy the asymptotic conditions as r and $s \to \infty$, the Legendre functions being defined by

$$Q_0(s) = \tfrac{1}{2}\log[(s+1)/(s-1)] , \qquad Q_1(s) = \tfrac{1}{2}s\log[(s+1)/(s-1)] - 1 ,$$

$$Q_2(s) = \tfrac{1}{4}(3s^2-1)\log[(s+1)/(s-1)] - \tfrac{3}{2}s , \qquad P_2(t) = \tfrac{1}{2}(3t^2-1) ,$$

and it being noted that

$$Q_0(s) \sim s^{-1} \sim r^{-1} , \qquad 3Q_1(s) \sim s^{-2} \sim r^{-2} , \qquad \tfrac{15}{2}Q_2(s) \sim s^{-3} \sim r^{-3} ,$$

as s and $r \to \infty$. To satisfy the boundary conditions on the body require

$$\left.\begin{aligned} \frac{\partial \chi}{\partial s} &= 3(k+m)Q_1(s_0)t^2 - s_0(t^2+1) , \\ \frac{\partial \chi}{\partial t} &= 3(k+m)Q_1(s_0)s_0 t - t(s_0^2+1) , \end{aligned}\right\} \tag{59}$$

on $s = s_0 = \cosh \alpha_0$. As for the oblate ellipsoid, equations (59) lead to three equations of simple form for the constants k, m and F. These equations are

$$\left.\begin{aligned} &k - m - 4F = 0 , \\ &k - \sigma_2 F = s_0/3Q_1(s_0) , \\ &m + \tau_2 F = 2/3\log[(s_0+1)/(s_0-1)] , \end{aligned}\right\} \tag{60}$$

The solution for k, the strength of the rotlet singularity in the far field asymptotic expansion of the velocity field, is

$$k = \tfrac{2}{3}(2s_0^2 - 1)[\tfrac{1}{2}(s_0^2 + 1)\log[(s_0 + 1)/(s_0 - 1)] - s_0]^{-1}, \tag{61}$$

and hence the torque acting on a prolate ellipsoid rotating with unit angular velocity about the y-axis is accordingly

$$\mathbf{G} = -\tfrac{16}{3}\pi\mu\hat{\mathbf{j}}(2s_0^2 - 1)[\tfrac{1}{2}(s_0^2 + 1)\log[(s_0 + 1)/(s_0 - 1)] - s_0]^{-1}. \tag{62}$$

Again this result may be recovered from Edwardes's general formula when the ellipsoid is prolate. The limiting case $s_0 \gg 1$, as for the oblate ellipsoid, corresponds to a large sphere of radius s_0 with centre at the origin. Equation (62) then gives $\mathbf{G} = -8\pi\mu s_0^3\hat{\mathbf{j}}$.

If the prolate ellipsoid rotates with angular velocity $\Omega\hat{\mathbf{j}}$ and the length of the minor semi-axis is a, the torque acting on the body is then $\mathbf{G} = -8\pi\mu\Omega k a^3\hat{\mathbf{j}}/(s_0^2 - 1)^{3/2}$.

The values of the two other constants m and F are found to be

$$m = \tfrac{2}{3}[\tfrac{1}{2}(s_0^2 + 1)\log[(s_0 + 1)/(s_0 - 1)] - s_0]^{-1}, \tag{63}$$

$$F = \tfrac{1}{3}(s_0^2 - 1)[\tfrac{1}{2}(s_0^2 + 1)\log[(s_0 + 1)/(s_0 - 1)] - s_0]^{-1}. \tag{64}$$

The constant m gives the strength of the stresslet singularity at infinity. The method presented in this paper leads to simple calculations for the rotlet and stresslet strengths k and m. Although k is directly related to the torque and hence is available from Edwardes's work, to determine m from Edwardes's solution it would be necessary to develop the far field asymptotic expansion of the solution appropriate for asymmetric rotation of an axisymmetric ellipsoid and then identify the stresslet term in that expansion. As far as we know, the constant m has not been determined before.

The general analysis presented in this paper is applicable to any axisymmetric body with fore-aft symmetry which rotates about a principal axis orthogonal to its axis of symmetry. By combining the torque calculated by our method for such flows with a calculation for the torque for axisymmetric rotation, the torque acting on the body when it rotates about *any* axis through its centre is then completely determined. Other body geometries for which the analysis of this paper is relevant include those considered for axisymmetric flow by Payne and Pell [13].

Appendix

Appendix I

The torque acting on a solid body bounded by a surface S for fluid motion exterior to the body is

$$\mathbf{G} = \int_S [\mathbf{r} \times \mathbf{R}_n]\, dS, \tag{A.1}$$

where $\mathbf{R}_n$ is the stress vector associated with an element of S with normal $\mathbf{n}$ drawn out of the body. If V is the volume of the region bounded by S and any surface Σ enclosing S, then

$$\int_\Sigma [\mathbf{r} \times \mathbf{R}_n]\, d\Sigma - \int_S [\mathbf{r} \times \mathbf{R}_n]\, dS = \int_V \frac{\partial}{\partial x_j}[\mathbf{r} \times \mathbf{R}_j]\, dV, \tag{A.2}$$

since $\mathbf{R}_n = \ell_j \mathbf{R}_j$, with $\mathbf{R}_j$ the stress vector for $\mathbf{n}$ coincident with $\hat{\mathbf{x}}_j$, and $\mathbf{n} = \ell_j \hat{\mathbf{x}}_j$. The right-hand side of (A.2) yields

$$\int_V [\hat{\mathbf{x}}_j \times \mathbf{R}_j] \, \mathrm{d}V + \int_V \left[\mathbf{r} \times \frac{\partial \mathbf{R}_j}{\partial x_j} \right] \mathrm{d}V .$$

But $\hat{\mathbf{x}}_j \times \mathbf{R}_j = 0$ and

$$\frac{\partial \mathbf{R}_j}{\partial x_j} = -\nabla p + \nabla^2 \mathbf{q} = 0 .$$

Thus

$$\mathbf{G} = \int_\Sigma [\mathbf{r} \times \mathbf{R}_n] \, \mathrm{d}\Sigma .$$

The surface Σ may be taken to be a sphere, centre at origin and radius R arbitrarily large. It is easy to show that the stresslet singularity gives rise to zero contribution to $\mathbf{G}$ identically and by letting $R \rightarrow \infty$, only the rotlet singularity gives a contribution to $\mathbf{G}$.

Appendix II

It will now be verified that

$$\mathbf{q} = \operatorname{curl}(C \hat{\boldsymbol{\phi}} \cos \phi) + \operatorname{grad}\left(\frac{B}{\rho} \cos \phi \right) + \operatorname{curl}\left(\frac{A}{\rho} \hat{\boldsymbol{k}} \sin \phi \right) \qquad \text{(B.1)}$$

is a solution of the Stokes equations

$$\nabla p = \mu \nabla^2 \mathbf{q} , \qquad \operatorname{div} \mathbf{q} = 0 , \qquad \text{(B.2)}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial}{\partial z^2} .$$

It is easily shown that

$$\nabla^2 \left(\frac{B}{\rho} \cos \phi \right) = \frac{L_{-1}(B)}{\rho} \cos \phi , \qquad \text{(B.3)}$$

$$\nabla^2 \left(\frac{A}{\rho} \sin \phi \right) = \frac{L_{-1}(A)}{\rho} \sin \phi , \qquad \text{(B.4)}$$

with

$$L_m \equiv \frac{\partial^2}{\partial \rho^2} + \frac{m}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} .$$

Consequently

$$\mathbf{q}_1 = \text{grad}\left(\frac{B}{\rho}\cos\phi\right) + \text{curl}\left(\frac{A}{\rho}\sin\phi\hat{\mathbf{k}}\right) \tag{B.5}$$

satisfies

$$\nabla^2\mathbf{q}_1 = 0\,, \qquad \text{div}\,\mathbf{q}_1 = 0 \tag{B.6}$$

if $L_{-1}(A) = L_{-1}(B) = 0$. The pressure generated by the velocity field $\mathbf{q}_1$ is therefore at most a constant.

Considering now

$$\mathbf{q}_2 = \text{curl}(C\hat{\boldsymbol{\phi}}\cos\phi)\,, \tag{B.7}$$

for which

$$\text{curl}\,\mathbf{q}_2 = \text{grad}\left(\frac{C}{\rho}\cos\phi\right) + \tfrac{1}{2}\nabla^2[C\sin 2\phi\hat{\mathbf{i}} - C(1+\cos 2\phi)\hat{\mathbf{j}}]\,.$$

Using (B.3), it follows that

$$\text{curl}\,\nabla^2\mathbf{q}_2 = \nabla^2\,\text{curl}\,\mathbf{q}_2 = 0 \tag{B.8}$$

if and only if C satisfies simultaneously the equations

$$L_{-1}(C) = 0\,, \qquad L_{-1}^2(C) = 0\,, \qquad (L_1 - 4/\rho^2)^2C = 0\,. \tag{B.9}$$

From the Weinstein decomposition formula [10], a solution of $L_1^2(C) = 0$ is

$$C = v^{(1)} + v^{(-1)}\,, \tag{B.10}$$

while a solution of $(L_1 - 4/\rho^2)^2C = 0$ is

$$\begin{aligned} C &= \rho^2[\omega^{(1)} + \omega^{(3)}] \\ &= \rho^2\omega^{(1)} + \omega^{(-1)}\,. \end{aligned} \tag{B.11}$$

Thus the only choice of a function C to satisfy all three of equations (B.9) is

$$C = v^{(-1)} \equiv \omega^{(-1)}\,. \tag{B.12}$$

This establishes that $\mathbf{q}_2$ given by (B.7) satisfies (B.8). Thus

$$\nabla^2\mathbf{q}_2 = \frac{1}{\mu}\nabla p$$

with p the pressure function, which can be verified to be of the form

$$p = -\frac{2\mu}{\rho}\frac{\partial C}{\partial z}\cos\phi\,. \tag{B.13}$$

By virtue of (B.6), p given by (B.13) represents the total pressure generated by the complete velocity field given by (B.1). Since this solution contains three independent functions A, B, C, it is a representation of the general solution of the Stokes equations appropriate for the variation in the azimuthal angle ϕ prescribed. It is equivalent to the general solution given by Lamb [12].

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